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NOTE ON A CLASS OF INTEGRAL EQUATIONS OF THE SECOND KIND.

BY CLYDE E. LOVE.

§ 1. **Introduction.** The object of the present paper is to develop the elementary theory of the singular integral equation

$$\varphi(x) = f(x) + \lambda \int_a^\infty K(x, t)\varphi(t)dt, \tag{1}$$

where $\varphi(x)$ is the unknown function, under the following assumptions: The function $K(x, t)$ is bounded and integrable in the square

$$I_b:\quad a \leqq x \leqq b, \qquad a \leqq t \leqq b,$$

where b is arbitrary, and is continuous in the region

$$I:\quad x \geqq a, \qquad t \geqq a,$$

or if discontinuous has only a finite number of discontinuities for any one value of x or of t; the function $f(x)$ is continuous in the interval

$$I:\quad x \geqq a;$$

the functions $f(x)$ and $K(x, t)$ may be written in the respective forms

$$f(x) = f_1(x)/x^\alpha,$$

$$K(x, t) = K_1(x, t)/x^\alpha t^\beta,$$

where $f(x)$ and $K(x, t)$ are bounded in I and in T respectively, and where α and β are real constants such that $\alpha + \beta > 1$.

The singular integral equation has been studied by Weyl* and Hobson† under hypotheses differing but slightly from those assumed above. While the results of the present note thus possess only a modicum of novelty, it is felt that the case here treated is of some interest because, as will presently appear, it can be solved by a direct extension of the method of Fredholm.

In what follows we shall assume $a > 0$. This restriction involves no loss of generality, since it may be removed by a mere translation of axes.

It will be convenient to put $\alpha + \beta = \gamma + 1$, whence $\gamma > 0$.

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* Math. Ann., Vol. 66 (1909), pp. 273–324.

† Proceedings of the London Math. Soc., Vol. 13 (1913), pp. 307–340.

§ 2. **Functions of the class** K. For brevity we shall say that, if a function satisfies all the conditions imposed above upon $K(x, t)$, that function *belongs to the class* K. We begin by noting certain properties of these functions.

First, if $K^{(1)}(x, t)$ is a function of the class K, the integral

$$\int_a^\infty |K^{(1)}(x, x)|\,dx$$

exists. For, the integral

$$\int_a^b K^{(1)}(x, x)dx$$

has a meaning for all values of $b > a$. Let $M^{(1)}$ denote the maximum value of $|K_1^{(1)}(x, t)|$ in the region T. Then, corresponding to every positive value of ϵ we can find a value of ξ such that

$$\int_\xi^{\xi'} |K^{(1)}(x, x)|\,dx < M^{(1)} \int_\xi^{\xi'} \frac{dx}{x^{\gamma+1}} < \frac{M^{(1)}}{\gamma \xi^\gamma} < \epsilon,$$

for all values of $\xi' > \xi$.

Further, if $K^{(2)}(x, t)$ is a second function of the class K, the integral

$$\overline{K}(x, t) = \int_a^\infty K^{(1)}(x, s)K^{(2)}(s, t)ds$$

converges absolutely for all values of x and t in the region T, represents a continuous function of x and t in T, and itself belongs to the class K.

To prove the first of these statements, we note that, the point (x, t) being any fixed point in T, the integral

$$\int_a^b K^{(1)}(x, s)K^{(2)}(s, t)ds$$

exists by hypothesis, and that to every ϵ there corresponds a ξ such that

$$\int_\xi^{\xi'} |K^{(1)}(x, s)K^{(2)}(s, t)|\,ds < \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta} \int_\xi^{\xi'} \frac{ds}{s^{\gamma+1}} < \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta} \cdot \frac{1}{\gamma \xi^\gamma} < \epsilon.$$

We shall prove the continuity of $\overline{K}(x, t)$ directly, by showing that to every positive ϵ there corresponds an η such that

$$|\overline{K}(x', t') - \overline{K}(x, t)| < \epsilon \tag{2}$$

whenever

$$|x - x'| < \eta, \qquad |t - t'| < \eta. \tag{3}$$

Having fixed ϵ, let us choose a number b satisfying the conditions

$$b > x, \qquad b > t,$$

$$b > \left(\frac{4M^{(1)}M^{(2)}}{a^{\gamma+1}\gamma\epsilon} \right)^{1/\gamma}. \tag{4}$$

Writing $\overline{K}(x, t)$ in the form

$$\overline{K}(x, t) = \int_a^b K^{(1)}(x, s)K^{(2)}(s, t)ds + \int_b^\infty K^{(1)}(x, s)K^{(2)}(s, t)ds,$$

we see that

$$|\overline{K}(x', t') - \overline{K}(x, t)| \leqq I_1 + I_2,$$

where

$$I_1 = \left| \int_a^b [K^{(1)}(x', s)K^{(2)}(s, t') - K^{(1)}(x, s)K^{(2)}(s, t)]ds \right|,$$

$$I_2 = \int_b^\infty \left[\frac{|K_1^{(1)}(x', s)K_1^{(2)}(s, t')|}{x'^\alpha t'^\beta} + \frac{|K_1^{(1)}(x, s)K_1^{(2)}(s, t)|}{x^\alpha t^\beta} \right] \frac{ds}{s^{\gamma+1}}.$$

Now it is well known* that the integral

$$\int_a^b K^{(1)}(x, s)K^{(2)}(s, t)ds$$

represents a continuous function of x and t in the region T_b; it follows at once that by suitable choice of η we may make

$$I_1 < \epsilon/2$$

for all values of x' and t' lying in T_b and satisfying the inequalities (3). As regards I_2, we have

$$I_2 \leqq M^{(1)}M^{(2)} \left[\frac{1}{x'^\alpha t'^\beta} + \frac{1}{x^\alpha t^\beta} \right] \int_b^\infty \frac{ds}{s^{\gamma+1}} < \frac{2M^{(1)}M^{(2)}}{a^{\gamma+1}\gamma b^\gamma} < \frac{\epsilon}{2},$$

by virtue of (4). Hence, if we place upon η the additional restrictions

$$\eta \leqq b - x, \qquad \eta \leqq b - t,$$

which is of course allowable, the relation (2) is established.

Finally, we note that

$$|\overline{K}(x, t)| \leqq \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta} \int_b^\infty \frac{ds}{s^{\gamma+1}} = \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta \gamma a^\gamma};$$

whence we may write

$$\overline{K}(x, t) = \overline{K}_1(x, t)/x^\alpha t^\beta,$$

where $\overline{K}_1(x, t)$ is bounded in T. This completes the proof that $\overline{K}(x, t)$ belongs to the class K.

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* Cf. Heywood et Fréchet, L'équation de Fredholm, p. 6.

§ 3. **The method of Fredholm.** Let us put

$$K\begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} = |K(x_i, t_j)|, \qquad i, j = 1, 2, \cdots, n$$

$$= \frac{K_1\begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix}}{(x_1x_2 \cdots x_n)^{\alpha}(t_1t_2 \cdots t_n)^{\beta}},$$

where

$$K_1\begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} = |K_1(x_i, t_j)|, \qquad i, j = 1, 2, \cdots, n;$$

also put

$$D(x, t, \lambda) = \sum_{n=0}^{\infty} u_n(x, t, \lambda), \tag{5}$$

$$D(\lambda) = \sum_{n=0}^{\infty} u_n(\lambda), \tag{6}$$

where

$$u_0(x, t, \lambda) = K(x, t),$$

$$u_n(x, t, \lambda) = \frac{(-\lambda)^n}{n!}\int_a^{\infty} \cdots \int_a^{\infty} K\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix} dx_1 \cdots dx_n, \qquad (n = 1, 2, \cdots)$$

$$u_0(\lambda) = 1,$$

$$u_n(\lambda) = \frac{(-\lambda)^n}{n!}\int_a^{\infty} \cdots \int_a^{\infty} K\begin{pmatrix} x_1, & \cdots, & x_n \\ x_1, & \cdots, & x_n \end{pmatrix} dx_1 \cdots dx_n \qquad (n = 1, 2, \cdots).$$

The convergence of the iterated integrals $u_n(x, t, \lambda)$, $u_n(\lambda)$ follows directly from the results of § 2. Further, denoting by M the maximum value of $|K_1(x, t)|$ in T, we have by Hadamard's theorem

$$\left| K_1\begin{pmatrix} x_1, & \cdots, & x_n \\ x_1, & \cdots, & x_n \end{pmatrix} \right| > \sqrt{n^n} M^n,$$

whence it follows very readily that series (6), and similarly (5), converges absolutely for all values of λ.

Thus in the present instance, just as in the case of the regular equation, the characteristic constants, or zeros of $D(\lambda)$, are isolated, and are determined by $K(x, t)$ alone. The " resolvent kernel "

$$K(x, t, \lambda) = D(x, t, \lambda)/D(\lambda) \tag{7}$$

is a meromorphic function of λ, and is defined as a function of x and t except when λ is a characteristic constant.

In what follows we confine our attention to the case where λ is not a characteristic constant.

§ 4. **Properties of the resolvent kernel.** It is easily seen that the function $K(x, t, \lambda)$ belongs to the class K. For, it follows directly from the convergence of series (5) that we may write

$$K(x, t, \lambda) = K_1(x, t, \lambda)/x^{\alpha}t^{\beta} \tag{8}$$

where $K_1(x, t, \lambda)$ is bounded in T. Further, the determinant

$$K\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix}$$

may evidently be written in the form

$$K\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix} = K(x, t)K\begin{pmatrix} x_1, \cdots, x_n \\ x_1, \cdots, x_n \end{pmatrix} + \Delta\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix},$$

where

$$\Delta\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix} = \begin{vmatrix} 0 & K(x, x_1) & \cdots & K(x, x_n) \\ K(x_1, t) & K(x_1, x_1) & \cdots & K(x_1, x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ K(x_n, t) & K(x_n, x_1) & \cdots & K(x_n, x_1) \end{vmatrix}.$$

Substituting in (5), we get

$$D(x, t, \lambda) = D(\lambda)K(x, t) + R(x, t, \lambda), \tag{9}$$

where

$$R(x, t, \lambda) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_a^{\infty} \cdots \int_a^{\infty} \Delta\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix} dx_1 \cdots dx_n. \tag{10}$$

It follows from § 2 that each term of the series (10) is a continuous function of x and t, and it is readily seen that the series converges uniformly for all values of x and t in T; hence $R(x, t, \lambda)$ is continuous in T. By (9),

$$K(x, t, \lambda) = K(x, t) + R(x, t, \lambda)/D(\lambda).$$

Since $K(x, t, \lambda)$ is equal to the sum of $K(x, t)$ and a continuous function, and has the form (8), it belongs to the class K.

It is now easy to establish the following fundamental formula:

$$\begin{aligned} K(x, t, \lambda) - K(x, t) &= \lambda \int_a^{\infty} K(x, s)K(s, t, \lambda)ds \\ &= \lambda \int_a^{\infty} K(x, s, \lambda)K(s, t)ds. \end{aligned} \tag{11}$$

We note first that, after expanding $\Delta\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix}$ by minors of the first row, we may write*

$$\int_a^\infty \cdots \int_a^\infty \Delta\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix} dx_1 \cdots dx_n$$

$$= -n\int_a^\infty \cdots \int_a^\infty \int_a^\infty K(x,\, s)K(s,\, t)K\begin{pmatrix} x_1, & \cdots, & x_{n-1} \\ x_1, & \cdots, & x_{n-1} \end{pmatrix} ds dx_1 \cdots dx_{n-1}$$

$$- n\int_a^\infty \cdots \int_a^\infty \int_a^\infty K(x,\, s)\Delta\begin{pmatrix} s, & x_1, & \cdots, & x_{n-1} \\ t, & x_1, & \cdots, & x_{n-1} \end{pmatrix} ds dx_1 \cdots dx_{n-1}.$$

Hence

$$R(x,\, t,\, \lambda) = R_1 + R_2,$$

where

$$R_1 = \lambda\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}}{(n-1)!}\int^\infty \cdots \int_a^\infty \int_a^\infty K(x,\, s)K(s,\, t)K\begin{pmatrix} x_1, & \cdots, & x_{n-1} \\ x_1, & \cdots, & x_{n-1} \end{pmatrix} ds dx_1 \cdots dx_{n-1}$$

$$= \lambda D(\lambda)\int_a^\infty K(x,\, s)K(s,\, t)ds,$$

and

$$R_2 = \lambda\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}}{(n-1)!}\int^\infty \cdots \int_a^\infty \int_a^\infty K(x,\, s)\Delta\begin{pmatrix} s, & x_1, & \cdots, & x_{n-1} \\ t, & x_1, & \cdots, & x_{n-1} \end{pmatrix} ds dx_1 \cdots dx_{n-1}.$$

Let us set

$$f_n(s) = \frac{\lambda(-\lambda)^{n-1}}{n!}\int_a^\infty \cdots \int_a^\infty K(x,\, s)\Delta\begin{pmatrix} s, & x_1, & \cdots, & x_{n-1} \\ t, & x_1, & \cdots, & x_{n-1} \end{pmatrix} dx_1 \cdots dx_{n-1},$$

and apply the following theorem:†

If the series $\sum_{n=1}^{\infty} f_n(s)$ converges uniformly in any fixed interval $(a,\, b)$, while the series

$$\sum_{n=1}^{\infty}\int_a^\infty f_n(s)ds$$

converges uniformly in an infinite interval $s \geqq a$, then we may write

$$\sum_{n=1}^{\infty}\int_a^\infty f_n(s)ds = \int^\infty \sum_{n=1}^{\infty} f_n(s)ds.$$

In the present instance, we have by Hadamard's theorem

$$|f_n(s)| < \frac{|\lambda|^n}{(n-1)!}\cdot\frac{\sqrt{n^n}M^{n+1}}{x^\alpha t^\beta s^{\gamma+1}}\int_a^\infty \cdots \int_a^\infty \frac{dx_1 \cdots dx_{n-1}}{(x_1 \cdots x_{n-1})^{\gamma+1}}$$

$$\leqq \frac{|\lambda|^n}{(n-1)!}\cdot\frac{\sqrt{n^n}M^{n+1}}{x^\alpha t^\beta a^{\gamma+1}}\cdot\frac{1}{\gamma^{n-1}a^{(n-1)\gamma}},$$

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* Cf. Heywood et Fréchet, loc. cit., p. 55.

† Bromwich, Infinite Series, p. 455.

whence the series $\sum_{n=1}^{\infty} f_n(s)$ converges uniformly for all values of $s \geqq a$. Further, we have

$$\left| \int_a^s f_n(s) ds \right| < \frac{|\lambda|^n}{(n-1)!} \cdot \frac{\sqrt{n^n} M^{n+1}}{x^\alpha t^\beta} \cdot \frac{1}{\gamma^n a^{n\gamma}},$$

so that the second condition of the theorem is satisfied. Therefore we may write

$$R_2 = \lambda \int_a^\infty K(x, s) \left[\sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}}{(n-1)!} \int_a^\infty \cdots \int_a^\infty \Delta \begin{pmatrix} s, x_1, \cdots, x_{n-1} \\ t, x_1, \cdots, x_{n-1} \end{pmatrix} dx_1 \cdots dx_{n-1} \right] ds$$

$$= \lambda \int_a^\infty K(x, s) R(s, t, \lambda) ds,$$

or, combining (12) and (13),

$$R(x, t, \lambda) = \lambda D(\lambda) \int_a^\infty K(x, s) K(s, t) ds + \lambda \int_a^\infty K(x, s) R(s, t, \lambda) ds.$$

Substituting for $R(x, t, \lambda)$ its value from (9), we get

$$D(x, t, \lambda) - D(\lambda) K(x, t) = \lambda \int_a^\infty K(x, s) D(s, t, \lambda) ds,$$

or, dividing by $D(\lambda)$,

$$K(x, t, \lambda) - K(x, t) = \lambda \int_a^\infty K(x, s) K(s, t, \lambda) ds.$$

Expanding $\Delta \begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix}$ by minors of the first column and proceeding similarly, we obtain the second desired formula,

$$K(x, t, \lambda) - K(x, t) = \lambda \int_a^\infty K(x, s, \lambda) K(s, t) ds.$$

§ 5. **The fundamental theorem.** We are now in position to prove the following

THEOREM: *If the functions $K(x, t)$ and $f(x)$ satisfy the conditions of* § 1, *and if λ is not a characteristic constant, the equation*

$$\varphi(x) = f(x) + \lambda \int_a^\infty K(x, t) \varphi(t) dt$$

has a unique continuous solution given by the formula

$$\varphi(x) = f(x) + \lambda \int_a^\infty K(x, t, \lambda) f(t) dt,$$

where $K(x, t, \lambda)$ is a meromorphic function of λ defined by equations (5), (6), (7). *This solution may be written in the form*

$$\varphi(x) = \varphi_1(x)/x^\alpha, \tag{14}$$

where $\varphi_1(x)$ is bounded in the interval $x \geqq a$.

First, assume that a solution of the form (14) exists.* Then

$$\varphi(t) = f(t) + \lambda \int_a^\infty K(t, t_1)\varphi(t_1)dt_1.$$

Multiplying by $K(x, t, \lambda)$ and integrating from $t = a$ to $t = \infty$, we find

$$\int_a^\infty K(x, t, \lambda)\varphi(t)dt = \int_a^\infty K(x, t, \lambda)f(t)dt + \lambda \int_a^\infty \int_a^\infty K(x, t, \lambda)K(t, t_1)\varphi(t_1)dt_1 dt;$$

it is easily shown that all the integrals here occurring exist. By means of (11)', this equation is transformed into

$$\int_a^\infty K(x, t, \lambda)\varphi(t)dt = \int_a^\infty K(x, t, \lambda)f(t)dt + \int_a^\infty [K(x, t, \lambda) - K(x, t)]\varphi(t)dt,$$

whence

$$\int_a^\infty K(x, t, \lambda)f(t)dt = \int_a^\infty K(x, t)\varphi(t)dt.$$

Substituting in (1) we get

$$\varphi(x) = f(x) + \lambda \int_a^\infty K(x, t, \lambda)f(t)dt. \tag{15}$$

It is evident at once that the function $\varphi(x)$ defined by (15) may be written in the form (14). For, let M_1 and M_2 denote the maximum values of $|K_1(x, t, \lambda)|$ and $|f_1(x)|$ respectively. Then

$$|\varphi(x)| \leqq \frac{M_2}{x^\alpha} + \frac{|\lambda| M_1 M_2}{x^\alpha} \int_a^\infty \frac{dt}{t^{\gamma+1}} = \frac{1}{x^\alpha}\left[M_2 + \frac{|\lambda| M_1 M_2}{\gamma a^\gamma}\right].$$

To see that the assumed solution exists, let us consider (15) as an integral equation in $f(x)$. The above argument, when applied to this equation, shows that the solution $f(x)$ is given by the formula

$$f(x) = \varphi(x) - \lambda \int_a^\infty K(x, t)\varphi(t)dt,$$

so that $\varphi(x)$ as given by (15) is a solution of (1).

Finally, assume that a second solution $\varphi_1(x)$ exists, so that

$$\varphi_1(x) = f(x) + \lambda \int_a^\infty K(x, t)\varphi_1(t)dt. \tag{16}$$

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* Cf. Heywood et Fréchet, loc. cit., p. 40.

Eliminating $f(x)$ between (15) and (16), we find

$$\varphi(x) - \varphi_1(x) = \lambda \int_a^\infty \varphi_1(t) \left[K(x, t, \lambda) - K(x, t) - \lambda \int_a^\infty K(x, s, \lambda) K(s, t) ds \right] dt,$$

or, by (11),

$$\varphi(x) - \varphi_1(x) = 0.$$

This completes the proof of the theorem.

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